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Optimal Tax Theory: A Synthesis

J. A. Mirrlees

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The main purposes of this paper are to unify some uncoordinated parts of the theory of optimal taxation, and to develop methods of analysis that can be quickly and easily applied to all kinds of optimal tax problems. The analysis is presented without attention to minor points of rigour (which I intend to treat elsewhere). As a result, the basic mathematical manipulations are relatively brief once the best ways of setting up the problems have been found. At the same time, a number of important details are treated in depth.

Theory can contribute to discussions about the levels of tax rates in a number of ways. It makes possible the calculation of optimal tax rates, to gain knowledge of how tax rates vary with objectives and possibilities. It shows how tax rates depend on certain indices (elasticities of various kinds, for example), thus indicating what form of evidence would be most useful and what influence that information would have on tax rates. It can formulate rules for optimal taxation which, though not expressed in terms of tax rates, may serve to attract attention to better measurements of the effects of economic policy. It can explore the consequences, for optimal tax rates or optimal taxation rules, of introducing considerations that have previously been absent or imperfectly present in models and policy discussions.

A number of papers have presented calculations of optimal tax rates under a variety of assumptions; including Atkinson (1973), Mirrlees (1971), and Stern (1976) for income taxation; Atkinson and Stiglitz (1972), and Deaton (1975) for commodity taxation. I contribute nothing to that topic in the present paper, which is concerned with formulae for optimal tax rates, and optimal taxation rules. Rules for commodity

taxation have been discussed in many papers, including Diamond and Mirrlees (1971), Mirrlees (1975), Ramsey (1927), and Stiglitz and Dasgupta (1971). Formulae involving optimal tax rates have been derived by Atkinson and Stiglitz (1972) for commodity taxes, and Mirrlees (1971) for a nonlinear income tax.

There is a striking contrast in this literature between the analysis and interpretation of first-order conditions which is usually emphasized in work on commodity taxation, and the development (and numerical implementation) of formulae for marginal tax rates that is characteristic of theories of optimal income taxation. This contrast seems to be quite basic to the theory of optimal taxation, as I shall argue in Section 1. Sections 1 and 2 are devoted to a relatively quick, and therefore nonrigorous, derivation of the main formal results in optimal tax theory, and a discussion of their interpretation. Section 3 outlines the conditions for a more general nonlinear theory.

The discussion of Section 2 raises the question of the interrelations between optimal commodity taxes and an optimal nonlinear tax on (say) labour incomes. In Section 4, first-order conditions for this case are derived, and discussed. This is the central part of the present paper. The analysis provides a unification of previous theories of optimal taxation, in that it covers linear (commodity) or fully nonlinear taxation, or any mixture of the two. This approach also enables one to derive a general 'Paretian' tax rule, corresponding to a result previously obtained (Mirrlees [1975]) for the two-class case. The rule is a necessary condition that must hold independently of the welfare function used. Naturally this rule is far from enough to determine the optimal tax

system. Using the whole set of first-order conditions, I discuss the implications of optimizing the labour-income tax for optimal commodity taxes, and vice versa. In particular, one would like to be able to gain some impression how the presence of commodity taxes bears upon the overall progression of the tax system, and on the progressivity of the labour-income tax. As yet, these implications are not entirely clear to me.

It is striking how easily the rules for the optimal provision of public goods can be obtained in the general setting introduced in Section 4; and how simple the rules are. They are derived in Section 5.

Finally in Section 6 it is shown how the various optimality conditions derived in the paper are particular instances of a general principle which is quite simple to formulate. This principle shows clearly how both efficiency and distributional considerations operate in the rules for optimal economic policy.

Apart from this 'fundamental principle', and the general 'duality' method used throughout, the most important particular new results of the paper are

- (1) The general conditions for optimal nonlinear taxes derived in Sections 2 and 3
- (2) A set of conditions in simple form, depending only on production prices and consumer preference orderings, which must necessarily hold for any tax equilibrium which is Pareto-efficient among the set of all tax equilibria. These conditions are stated at their simplest in equation (2.22).



(3) The conditions for optimal linear taxes in the presence of optimal nonlinear taxes, derived in Section 4. These provide a strikingly simple criterion for assessing the optimal impact of commodity taxation when there is an income tax.

(4) The form of the condition for optimal income taxation in the presence of commodity taxes (optimal or not), also presented in Section 4.

(5) The conditions for optimal provision of public goods in the presence of an optimal income tax are given a neat and illuminating form, which has an interesting connection with the commodity-tax conditions just mentioned.

#### 1 Optimal Taxation in the Linear Case

In the theory of optimal commodity taxation, it has been usual to consider a finite population of households, whose net demands will here be denoted by vectors:

$$x^1, x^2, x^3, \dots, x^H.$$

These households face prices  $q$  for commodities, with

$$q = p + t$$

where  $p$  are producer prices, and  $t$  are tax rates. Thus each household has the same budget constraint  $q \cdot x \leq 0$ . In the present paper, I assume that:

- (i) Production available to consumers is constrained by a production constraint

$$p \cdot \sum_h x^h = A \quad (1.1)$$

with  $p$  and  $A$  constant. This is not a serious restriction. The linear constraint can be thought of as a linear approximation to production possibilities in the neighborhood of the optimum, in which case the producer prices  $p$  are to be regarded as marginal costs and marginal products. So long as first-order necessary conditions are at issue, it does not matter that  $p$  is constant. A fuller discussion of production is given in (Diamond and Mirrlees, 1971).

- (ii) Pure profits, if any, are paid to the State. This excludes certain interesting issues that have been discussed in the literature, but allows us to concentrate on the main taxation issues without having to deal with producer taxation alongside consumer taxation.

- (iii) An individualistic welfare function

$$W = \sum_h u^h(x^h) \quad (1.2)$$

is to be maximized. The representation (1.2) is implied by complete separability of consumption by individuals, but is not necessary for most of the results of this paper, since only local changes are considered. (1.2) simply represents a convenient choice of utility functions representing individual preferences so that the marginal weights attached to utility levels are all equal.

The first-order conditions for optimal choice of the tax rates  $t$  (or, equivalently  $q$ ) will be derived by duality methods. Let

$$m^h(q, u) \quad (1.3)$$

be the expenditure function for household  $h$ , i.e., the minimum expenditure required to attain utility  $u$  when prices are  $q$ . Then the (Slutsky) compensated demand functions for household  $h$  are

$$x^{ch}(q, u) = m_q^h \quad (1.4)$$

where the subscript here and subsequently denotes differentiation.

The utility level of household  $h$  will be denoted by  $v^h$ .

Then the optimal taxation problem can be expressed as

$$\begin{aligned} & p \cdot \sum_h x^{ch}(q, v^h) = A, \\ \text{Max}_{q, h} \sum_h v^h : & \\ & m^h(q, v^h) = 0 \quad (\text{all } h) \end{aligned} \quad (1.5)$$

The second set of constraints here reflects the assumption that households have no lump-sum income or expenditure.

It is of the first importance to realise why the optimal commodity tax problem is best set without explicit reference to tax rates. Suppose  $q^*$  is the solution to the problem. Then  $t^* = q^* - p$  are optimal tax rates. But the real equilibrium of the economy is unaltered if instead we have tax rates

$$t_\rho^* = \rho q^* - p \quad (1.6)$$

for any positive constant  $\rho$ . It follows that (except when no taxation is optimal) there is no answer to the question: which commodities should be taxes and which subsidised? The 'answer' varies with the arbitrary

choice of a number  $\rho$ . A specific answer can be given if it is stipulated that some particular commodity is to have a zero tax rate; but that is usually an uninteresting device because there is no naturally untaxed commodity.

For this reason, one should not seek formulae for optimal tax rates in the general problem. One can hope to find equations identifying commodities whose net demand is encouraged, or discouraged, by the tax system, since these are real properties of the tax system, which remain invariant under the trivial change in tax rates described by (1.6). Having set the problem up as in (1.5) we shall find that such conditions appear almost immediately. The other advantage of this new way of setting up the problem is that it is more easily related to the problem of nonlinear taxation.

To obtain necessary conditions for the optimum in (1.5), introduce Lagrange multipliers for the constraints, and set the derivatives of the following expression with respect to  $v^1, \dots, v^H, q$  (holding  $p$  constant) equal to zero:

$$L = \sum_h v^h - \lambda p \cdot \sum_h x^{ch}(q, v^h) - \sum_h \mu_h m^h(q, v^h). \quad (1.7)$$

Differentiating  $L$  with respect to  $v^h$ , we obtain

$$1 = \lambda p \cdot x_v^{ch} + \mu_h m_v^h. \quad (1.8)$$

Differentiation with respect to  $q$ , with  $p$  and the  $v^h$  constant, yields

$$\lambda p \cdot \sum_h x_q^{ch} + \sum_h \mu_h m_q^h = 0. \quad (1.9)$$

Using (1.4), and the symmetry of the Slutsky derivatives, we obtain from (1.9)

$$-\lambda \sum_h x_q^{ch} \cdot p = \sum_h \mu_h x_h^{ch}. \quad (1.10)$$

Now  $x_q^{ch} \cdot q = 0$ , by homogeneity. Therefore

$$\begin{aligned} -x_q^{ch} \cdot p &= x_q^{ch} \cdot (q - p) = x_q^{ch} \cdot t \\ &= \frac{\partial}{\partial \theta} x^{ch}(p + \theta t) \Big|_{\theta=1}; \end{aligned} \quad (1.11)$$

and we can write (1.10) in the form

$$\sum_h x_q^{ch} \cdot t = \frac{\partial}{\partial \theta} \sum_h x^{ch}(p + \theta t) \Big|_{\theta=1} = \sum (\mu_h / \lambda) x_h^{ch}. \quad (1.12)$$

In words, the total substitution effects of a proportional change in all tax and subsidy rates should be proportional to a weighted sum of demands. We shall see that  $\mu_h$  can be negative for some  $h$ . The weights in (1.12) are obtained from (1.8):

$$\begin{aligned} \frac{\mu_h}{\lambda} &= (\lambda^{-1} - p \cdot x_v^{ch}) / m_v^h = 1/(\lambda m_v^h) - p \cdot x_m^h(q, 0) \\ &= \frac{1}{\lambda m_v^h} + t \cdot x_m^h(q, 0) - 1 \end{aligned} \quad (1.13)$$

$x^h$  is the uncompensated demand function, and  $x_m^h$  its derivative with respect to income. In words, the weight is the difference of a term proportional to the marginal utility of income, and the income-derivative of the household's income net of taxes.

The above conditions were derived on the assumption that there is no lump-sum taxation. It is always possible to have lump-sum taxation (or subsidization) at a uniform rate. If this is done, the constraints  $m^h = 0$  should be replaced by the constraint  $m^h = b$ , with the subsidy



b free to be chosen. The analysis is then unchanged, except that differentiation of the Lagrangean with respect to b yields a further condition

$$\Sigma \mu_h = 0 \quad (1.14)$$

$\lambda$  and  $\mu_h$  were introduced as multipliers dual to the government's revenue constraint (which is equivalent to (1.1) for given household incomes) and the constraints on household income. Thus  $\lambda$  is the social marginal utility of income transferred to household h, the production constraint being unchanged. Since the production constraint is  $\Sigma m^h - \text{taxes} = A$ , the increase in income involves a tax change. The social marginal utility of income, ignoring this tax change, is  $\lambda + \mu_h$ . Diamond (1975) calls this the social marginal utility of income.  $\mu_h$  can be termed the social marginal utility of transfer.

The form of the first-order conditions (1.12) differs from those derived in (Diamond and Mirrlees, 1971). The other form is readily derived from (1.8) and (1.9) by eliminating  $\mu_h$ : after some manipulation one has

$$\lambda \frac{\partial}{\partial t} \Sigma t \cdot x^h(p + t, m) = \Sigma x^h / m_v^h. \quad (1.15)$$

In words, the revenue effects of tax changes should be proportional to welfare-weighted demands.

Of the two forms, (1.12) and (1.15), for the first-order conditions, we shall see that (1.12) corresponds best to the conditions for optimal nonlinear taxation.

Equation (1.12) is appealing chiefly because the left hand side,  $\Sigma x_q^{ch} \cdot t$ , is a measure of the extent to which the tax system discourages a commodity. If the tax system is intensified, in the sense that all taxes

and subsidies are changed proportionately, demands for commodities are subject to both income and substitution effects. The income effects,  $-x_m^h x^h \cdot t$ , are simply the income-derivatives of demand times taxes paid. The total substitution effect is  $\Sigma x_q^{ch} \cdot t$  (as can be seen from (1.12)). Since the income effect has nothing to do with the way the tax system bears on commodities individually, it should be ignored in assessing the effects of optimal taxes. Thus, we can reasonably introduce the definition:-

Under a tax system, the index of discouragement of commodity  $i$ ,

$$d_i = \Sigma_{hj} \frac{\partial x_i^{ch}}{\partial q_j} t_j / \Sigma x_i^h \quad (1.16)$$

The first order conditions for optimal commodity taxation can then be expressed as

$$d_i = \frac{\Sigma (\mu_h / \lambda) x_i^h}{\Sigma x_i^h} \quad (1.17)$$

where  $\mu_h / \lambda$  is given by (1.13), and satisfies  $\Sigma (\mu_h / \lambda) = 0$ .

## 2 Optimal Nonlinear Taxation

Linearity of the tax system plays an essential role in the previous section, because linearity defines control variables  $(q)$  for the optimization problem. With nonlinear taxation, the government can, in effect, impose any budget constraint it wishes on the members of population, the only restriction being that the constraint is the same for everyone. When policy options are extended in this way, it is best to consider a

continuum population of households. Then one may avoid having as optimum policy an awkwardly shaped budget set with many corners. It is very restrictive, despite appearances, to describe the population by a single parameter  $n$ . Nevertheless we must usually do so to get neat results. The analysis will be carried out for the one-parameter case,  $n$  being distributed with density function  $f(n)$ . The many-parameter case is outlined in Section 3.

For the nonlinear case, it is useful to identify a numéraire commodity. The notation is now that household  $n$ , with utility function  $u(x,z,n)$ , chooses a vector  $x(n)$  of net demands for nonnuméraire goods and a net demand  $z(n)$  for numéraire.  $u$  is a nondecreasing function of all variables. We have to consider allocations in which

$$x(n), z(n) \text{ maximizes } u(x, z, n) \text{ subject to } (x, z) \in B \quad (2.1)$$

where  $B$  is the budget set, describing the effect of taxes and subsidies in combination with producer prices. It should be noted that the consumption set, of net demands feasible for the consumer, may vary with  $n$ . This will be ignored in the following, where the set of definition of  $u$  is supposed the same for all  $n$ .

To reduce (2.1) to a more manageable form, the method of Mirrlees (1971) is generalized. The idea is that, if we define

$$v(n) = u(x(n), z(n), n), \quad (2.2)$$

(2.1) implies that

$$v(n) - u(x(n), z(n), n) = 0 \leq v(m) - u(x(n), z(n), m), \quad (2.3)$$

i.e.,  $m = n$  minimizes  $v(m) - u(x(n), z(n), m)$ . Consequently,

$$v'(n) = u_n(x(n), z(n), n) \quad (2.4)$$

When  $x$  and  $z$  are differentiable with respect to  $n$ , differentiation of (2.2) shows that (2.4) is equivalent to

$$u_x \cdot x'(n) + u_z z'(n) = 0.$$

which we can write more neatly, if we introduce the marginal rate of substitution

$$s(x, z, n) = u_x / u_z, \quad (2.5)$$

as

$$z'(n) + s(x(n), z(n), n) \cdot x'(n) = 0 \quad (2.6)$$

We also have a second order condition from the minimization of  $v - u$ :

$$v''(n) \geq u_{nn}(x(n), z(n), n). \quad (2.7)$$

This is equivalent (when  $x$  and  $z$  are differentiable) to

$$s_n(x(n), z(n), n) \cdot x'(n) \geq 0 \quad (2.8)$$

To deduce (2.8) from (2.7), differentiate (2.4) with respect to  $n$ :

$$v''(n) = u_{nx} \cdot x'(n) + u_{nz} z'(n) + u_{nn}$$

Applying (2.5), we have

$$u_{nx} \cdot x'(n) + u_{nz} z'(n) \geq 0 \quad (2.9)$$

(2.6) allows us to replace  $z'(n)$  in this inequality:

$$u_{nx} - u_{nz} \frac{u_x}{u_z} \cdot x'(n) \geq 0. \quad (2.10)$$

Since

$$\frac{\partial s}{\partial n} = \frac{u_{nx}}{u_z} - \frac{u_{nz} u_x}{u_z^2},$$

(2.10) is in turn equivalent to (2.8) (since  $u_z > 0$  by assumption).

In an appendix to this paper, it is shown that (2.4) in conjunction with a condition a little stronger than (2.8),

$$s_m(x(n), z(n), m) \cdot x'(n) \geq 0, \text{ all } m, n \quad (2.11)$$

together imply that (2.1) holds for some budget set  $B$ . When there are only two commodities, this is a very satisfactory situation. Then  $s_n > 0$  is a sufficient assumption to ensure that (2.1) is equivalent to (2.4) and  $z'(n) \geq 0$ . It is this circumstance that makes the income-tax problem studied in Mirrlees (1971) manageable.

In the many-commodity case, we can replace the consumer constraint (2.1) by the weaker condition (2.4). If the solution of that problem satisfies (2.11), then it is a solution of the basic problem. This gives us an easy check on a computed solution, but it is perfectly possible that  $s_n \cdot x'(n)$  will vanish for certain ranges of  $n$  in the optimum. In that case a more detailed analysis than the one below would be required.

We shall now obtain necessary conditions for the problem

$$\begin{aligned} \int [p \cdot x(n) + z(n)] f(n) dn &= A \\ \text{Max } \int v(n) f(n) dn : & \\ v'(n) &= u_n(x(n), z(n), n) \end{aligned} \quad (2.12)$$

Notice that production prices are based on  $z$  as numéraire. If labour is numéraire,  $z(n)$  should be a negative number, being supplied by households, not demanded. Recollect that

$$v(n) = u(x(n), z(n), n) \quad (2.13)$$



and  $f$  is the density function describing the composition of the population.  $n$  is assumed distributed between 0 and  $\infty$ . In what follows, subscripts denote partial derivatives.

Having formulated the problem in this way, it will obviously pay to invert the utility function, writing

$$z(n) = \zeta(x(n), v(n), n). \quad (2.14)$$

In effect this transformation is a 'duality trick', as we shall see in Section 4. Since (2.14) is derived from (2.13), we can calculate the derivatives

$$\zeta_x = -u_x/u_z = -s, \quad (2.15)$$

$$\zeta_v = 1/u_z$$

Converting (2.12) into Lagrangean form we set equal to zero the derivatives of

$$L = \int [(v - \lambda p \cdot x - \lambda \zeta)f + \mu v'(n) - \mu u_n(x, \zeta, n)] dn.$$

Differentiating in turn with respect to  $v(\cdot)$  and  $x(\cdot)$ , we obtain

(noting that  $u_{nx} + u_{nz} \cdot (-s) = u_{zs}$ )

$$(1 - \frac{\lambda}{u_z})f - \mu \frac{u_{nz}}{u_z} - \mu'(n) = 0 \quad (2.16)$$

$$\lambda(s - p)f = \mu u_{zs}. \quad (2.17)$$

The last term in (2.16) is obtained by first integrating by parts the term  $\mu v'$  in  $L$ . This leaves an integrated part  $\lim_{n \rightarrow \infty} \mu(n)v(n) - \mu(0)v(0)$ . Differentiation with respect to  $v(0)$  and  $v(\infty)$  (this can be checked by a more careful analysis), yields the further conditions

$$\mu(0) = 0, \quad \mu(\infty) = 0. \quad (2.18)$$

Equation (2.17) holds only when  $(x(n), z(n))$  is not up against the consumer's supply constraint. For example the possibility of a zero labour supply is not allowed. Though this point must be considered for numerical calculations, it will be ignored here as not being of great importance.

The first, simplest and most exciting, feature of these first-order conditions is the set of equations (2.17), which are statements about marginal tax rates. If the numéraire is untaxed, and  $\tau_i$  is the marginal tax rate on commodity  $i$ , the marginal price facing the consumer is  $p_i + \tau_i$ , which he will equate to  $s_i$ , the marginal rate of substitution between that commodity and numéraire. (2.17) says that

$$\tau_i = \frac{v}{f} \frac{\partial s_i}{\partial n}, \quad v = \mu u_z / \lambda \quad (2.19)$$

To fix ideas for the moment, let  $n$  describe ability or willingness to work. Mathematically, this says that as  $n$  increases, individuals find it easier to do additional work of the same productive value, so that, for most goods  $s_i$  should be an increasing function of  $n$ . The exceptional goods are those whose marginal utility falls faster than the marginal disutility of labour. (Remember that labour is measured not in units of time but in units of equal marginal productivity.) Normally it is thought right that the tax system should bear more heavily on the more able. Thus marginal tax rates are expected to be positive on most commodities. This being so, the numerical factor in (2.19) ought normally to be positive. In that case, our condition says that marginal taxes should be greater on commodities the more able would tend to prefer.

More explicitly, (2.19) implies that

$$\frac{\tau_i}{s_i} - \frac{\tau_j}{s_j} = \frac{v}{f} \frac{\partial}{\partial n} (s_i/s_j), \quad v = \mu u_z/\lambda \quad (2.20)$$

So that the question which commodity should be taxed more highly is answered by reference to the effect of an increase in  $n$  upon the slope of the individual's indifference curves.

This prescription is most agreeable to common-sense; but it should be remembered that it is not true when only linear taxation is allowed. In that case, as we saw, more awkward statements about the effect of general tax changes on aggregate compensated demands must be made. It is the extra scope for policy allowed by nonlinearity that restores the 'common-sense' result.

Perhaps the most surprising feature of (2.17) is the strong implications that are independent of the welfare function assumed, or even the population distribution, namely (dividing the equations for different commodities)

$$\frac{s_i - p_i}{s_j - p_j} = \frac{s_{in}}{s_{jn}} \quad (2.21)$$

Whether these equations hold can be tested by reference to indifference curves alone (and the way they vary with  $n$ ). Therefore these equations must hold (for all  $i$  and  $j$ ) as a condition of Pareto-efficiency. If they do not hold, there exists a tax system which would make everyone better off.

Since  $\frac{\partial}{\partial n} \tau_i = \frac{\partial}{\partial n} (s_i - p_i) = s_{in}$ , (2.21) takes an even neater form:

$$\frac{\partial}{\partial n} \frac{\tau_i}{\tau_j} = 0 \quad (2.22)$$

Thus it is necessary for Pareto-efficiency (relative to the set of tax equilibria) that the ratios of marginal tax rates are locally independent of  $n$ . (2.22) is simple but mysterious. The  $\tau_i$  are to be regarded as functions of  $x$ ,  $z$  and  $n$ . We ask what  $\tau'$  would make an  $n$ -man choose the demands of an  $(n + dn)$ -man. The ratios of these  $\tau'$  have to be the same as the ratios of the marginal tax rates the  $(n + dn)$ -man actually faces. This is a strong requirement. No intuitive explanation of the result has occurred to me.

Returning to equation (2.20), we see that the marginal tax rates on commodities  $i$  and  $j$  should be the same for all individuals if  $s_i/s_j$  is independent of  $n$ . This means that  $x_i$  and  $x_j$  enter the utility function through a subutility function which is the same for everyone. More generally, if a subset of commodities, represented by a vector  $x^1$ , enters the utility function in the following way,

$$u = U(a_1(x^1), x^2, z, n) \quad (2.23)$$

all the commodities in  $x^1$  should be taxed at the same rate. In particular, when utility has this form with no commodities in the second group, it is optimal to have a tax schedule for the numéraire alone, with no taxes on other commodities. This notable result generalizes that obtained by Atkinson and Stiglitz (1976) for the additively separable case. It is true also when  $n$  is multi-dimensional.

The remaining task, in interpreting the optimality conditions, is to use equation (2.16) to get more precise information about the sign and magnitude of the multipliers  $\mu(n)$ . Let us first, to encourage intuition, assign a special symbol to the marginal utility of numéraire

(e.g., the marginal disutility of labour-value) which plays in the model much the role of the marginal utility of income in one- and two-good models:

$$\beta = u_z \quad (2.24)$$

$\beta$  is to be regarded as a function of  $x$ ,  $z$  and  $n$ ; and  $\beta_n$ , for example, means the partial derivative of  $\beta$  with respect to  $n$ ; but  $\beta(n)$  and  $\beta_n(n)$  mean  $\beta(x(n), z(n), n)$  and  $\beta_n(x(n), z(n), n)$  respectively.

There are two convenient ways of integrating (2.16) for  $\mu$ . Integrating as it stands, and using the boundary condition  $\mu(0) = 0$ , we get

$$\mu(n) = \int_0^n \left[1 - \frac{\lambda}{\beta(m)}\right] \exp\left[\int_n^m \beta_n(m')/\beta(m') \cdot dm'\right] f(m) dm \quad (2.25)$$

The alternative equation for  $\mu$  is obtained by rewriting (2.16) in the form

$$\frac{d}{dn} (\beta\mu) - s_z \cdot x'(n) \cdot \beta\mu - (\beta - \lambda)f = 0 \quad (2.26)$$

The derivation of (2.26) is based on the calculation

$$\begin{aligned} \frac{d}{dn} \beta &= \beta_n + \beta_x \cdot x'(n) + \beta_z z'(n) \\ &= \beta_n + u_{zx} \cdot x' + u_{zz} (-u_x \cdot x') \\ &= \beta_n + u_z \frac{\partial}{\partial z} (u_x/u_z) \cdot x' \\ &= \beta_n + \beta s_z \cdot x' \end{aligned} \quad (2.27)$$

Integration of (2.26) yields the formula

$$\beta\mu = \int_0^n [\beta(m) - \lambda] \exp\left[\int_m^n s_z \cdot x'(m') dm'\right] f(m) dm \quad (2.28)$$



(2.25) and (2.28) suggest two special cases that should be easy to handle, namely  $\beta_n \equiv 0$  (which simplifies (2.25)) and  $s_z \equiv 0$  (which simplifies (2.28)). Take the latter case.  $s$  is independent of  $z$  if and only if utility can be expressed in the form.

$$u = U(a(x,n) + z,n) \quad (2.29)$$

This is the case where income effects vanish in ordinary consumer theory.

If utility were to take this form, we should have

$$\beta\mu = \int_0^n [\beta(m) - \lambda]f(m)dm \quad (2.30)$$

and, from (2.27),  $\frac{d}{dn} \beta = \beta_n$ . Consequently  $\beta_n \leq 0$  implies that  $\beta(m)$  is nondecreasing. Let  $\beta(n_1) = \lambda$ . Then (2.30) implies that  $\mu \geq 0$  when  $n \leq n_1$ . But  $\mu = 0$  when  $n = \infty$ . Therefore  $\beta\mu$  is equal to  $-\int_n^\infty [\beta(m) - \lambda]f(m)dm$ ; and it follows that  $\mu \geq 0$  for  $n \geq n_1$  also.

What we have found is this. In the no-income-effects case,

$$\beta_n \leq 0 \quad (2.31)$$

is a sufficient condition for

$$\mu(n) \geq 0 \quad (2.32)$$

It will be recollected that the tax rules (2.17) then have their natural interpretation, for  $\lambda$ ,  $f$ , and  $u_z = \beta$  are all nonnegative.

Condition (2.31), stating that the marginal utility of numéraire is a decreasing function of  $n$ , for fixed  $x$  and  $z$ , expresses the assumption that consumers with larger  $n$  are less deserving. We have now shown (cf. the discussion of equation (2.19) above) that in the no-income-effect case, it is indeed implied that the tax system should

bear more heavily on those with greater ability. But in general, it seems that one cannot establish the result; for the last term in equation (2.27),  $\beta s_z \cdot x'$ , is quite likely to be positive. There are other special cases, notably the two-commodity case with normality, for which a theorem can be established (cf. Mirrlees (1971)). Also, cases where  $\mu$  is negative for some  $n$  are odd in other respects. Consider (2.17) again,

$$\lambda(s - p)f = \mu u_z s_n$$

and multiply by  $x'$ :

$$(s - p) \cdot x' = \frac{\mu u_z}{\lambda f} s_n \cdot x' \quad (2.33)$$

We know that  $s_n \cdot x' \geq 0$  (equation (2.8)); and that  $s \cdot x' = -z'$  (2.6). Therefore

$$\mu < 0 \text{ implies } p \cdot x' + z' \geq 0 \quad (2.34)$$

If the numéraire is not taxed,  $-z$  is the consumer's expenditure, and the total tax paid by an  $n$ -man is

$$T(n) = -z(n) - p \cdot x(n). \quad (2.35)$$

It follows that

$$\mu < 0 \text{ implies } T'(n) \leq 0. \quad (2.36)$$

In words,  $\mu$  is negative only if the tax system does not bear more heavily on consumers with greater  $n$ .

A number of economists have examined, usually for special cases, the implications for marginal tax rates of the boundary conditions that  $\mu = 0$  at the ends of the skill distribution. Equation (2.17) suggests

that marginal tax rates should be zero when  $n = 0$  or  $\infty$ , provided that  $f$  does not tend to zero too rapidly. Phelps (1974) and Sadka (1974) have followed this idea for  $n = \infty$ , and more recently Seade (1975) has noticed a similar result for the lower end of the distribution. When the result holds at both ends, the tax system is neither progressive for all  $n$ , nor regressive. Calculations suggest to me that these end results are of little practical value. When the conditions for their validity hold, it is usually true that zero is a bad approximation to the marginal tax rate even within most of the top and bottom percentiles.

### 3 Multiple Characteristics

In the previous section, the population was described by a single parameter. The power of this assumption is shown in the conclusion that marginal tax rates are proportional to  $\partial s_i / \partial n$ . Matters are not quite so simple when the population is described by a vector  $n$ . Let us suppose there are  $I + 1$  commodities, and  $J$  parameters describing the population.

It will be recollected that, in the one-parameter case, consumer utility-maximization implies that

$$v'(n) = u_n(x(n), z(n), n). \quad (3.1)$$

The argument leading to this conclusion is valid also for many parameters, and we have therefore

$$\frac{\partial v}{\partial n_j} = u_{n_j}(x(n), z(n), n) \quad (3.2)$$

for each parameter. Thus we can read (3.1) as a vector equation. Similarly we obtain from second-order conditions the implication that the matrix

$$\left[ \frac{\partial x}{\partial n_j} \cdot s_{n_k}(x(n), z(n), n) \right] \text{ is nonnegative definite.} \quad (3.3)$$

(This matrix is in fact symmetric, as one can deduce by differentiating (3.1) with respect to  $n$ .) The question then arises whether, as in the one-parameter case of (3.1), in conjunction with a condition similar to (3.3), implies utility maximization subject to some budget constraint. Fortunately it does so, as I shall establish elsewhere. This allows an argument that in a large class of cases the constraint (3.3), or rather a stronger form analogous to (2.11) can be neglected in deriving the first-order conditions for constrained maximization.

In that case, we have to consider a Lagrangean

$$L = \int [\{v(n) - \lambda p \cdot x(n) - \lambda s\}f + \{v'(n) - u_n(x, \zeta, n)\} \cdot \mu(n)] dn \quad (3.4)$$

where integration is over the whole nonnegative orthant in the space of parameter vectors  $n$ , and  $\mu(n)$  is, for each  $n$ , a vector of multipliers commensurate with  $n$ . As before,  $\zeta$  is the function of  $x$ ,  $v$  and  $n$  defined by  $v = u(x, \zeta, n)$ . Before differentiating  $L$ , we want to replace the term

$$\int v'(n) \cdot \mu(n) dn = \int \dots \int \sum_j \frac{\partial v}{\partial n_j} \mu_j(n) dn_1 \dots dn_J$$

by something more convenient. This can be done by using a standard theorem in multidimensional calculus, which states that, for nice functions  $v, \mu_1, \dots, \mu_J$  defined in a closed region  $D$

$$\int_D \sum_j \frac{\partial v}{\partial n_j} \mu_j dn + \int_D v \sum_j \frac{\partial \mu_j}{\partial n_j} dn = \int_{\partial D} v \mu \cdot ds \quad (3.5)$$

where  $\partial D$  is the boundary of  $D$ , and  $ds$  is outward normal to this surface. This theorem allows us to write

$$L = \int [ \{v - \lambda p \cdot x - \lambda \zeta\} f - v \nabla \cdot \mu - u_n \cdot \mu ] dn + \int_{\partial D} v \mu \cdot ds \quad (3.6)$$

with  $\partial D$  the boundary of the nonnegative orthant, and  $\nabla \cdot \mu$  a standard notation for the divergence  $\sum_j \partial \mu_j / \partial n_j$ .

Setting the derivatives of  $L$  with respect to  $x(n)$  and  $v(n)$  equal to zero as in the one-dimensional case, we get

$$(s - p)f = \frac{u_z}{\lambda} s_n \cdot \mu \quad (3.7)$$

$$\nabla \cdot \mu + \frac{u_{zn}}{u_z} \cdot \mu = (1 - \frac{\lambda}{u_z})f \quad (3.8)$$

(Note that  $s_n \cdot \mu$  is a vector with components  $\sum_j (\partial s_i / \partial n_j) \mu_j$ , and  $u_{zn} \cdot \mu$  is defined similarly.) These equations are supplemented by boundary conditions on  $\mu$  obtained by considering variations of  $v$  on the boundary of the orthant:

$$\mu_j = 0 \quad \text{when} \quad n_j = 0, \infty \quad (3.9)$$

No doubt a thorough analysis of these conditions would be quite complicated. The basic ideas are tolerably clear. First one should consider the system of equations

$$\begin{aligned} v &= u(x, z, n) \\ v' &= u_n(x, z, n) \end{aligned} \quad (3.10)$$

$$(s - p)f = \frac{u_z}{\lambda} s_n \cdot \mu$$

There are  $I + J + 1$  of these equations. Consequently one can hope to solve for  $J$  variables  $\mu_j / \lambda$ , the  $I$  variables  $x_i$ , and  $z$ , as functions of  $v, v' = (\partial v / \partial n_1, \dots, \partial f / \partial n_J)$ , and  $n$ . Substituting in (3.8), one has

$$\sum \frac{\partial}{\partial n_j} \mu_j(v, v', n) + \sum \frac{u_{zn_j}(x(v, v', n), z(v, v', n), n)}{u_z} \mu_j(v, v', n) = (1 - \frac{\lambda}{u_z})f. \quad (3.11)$$

This is a second-order partial differential equation in  $v$ . It is to be associated with boundary conditions upon  $v$  and  $v'$  on the boundary of the nonnegative orthants which are defined by (3.9). For this to make sense, (3.11) should be an elliptic equation. Methods for computing the solution of such partial differential equations with given boundary conditions are available. Once (3.11) has been solved for  $v$ ,  $x(n)$  and  $z(n)$  can be obtained from (3.10), thus defining the desired budget set.

It is interesting to note that the budget set so obtained will in general be of dimension  $\min(I, J)$ . So long as  $J \leq I$ , there is no reason why consumers of different types should not be choosing different consumption plans in the optimal equilibrium. Then the budget set defined as

$$B = \{(x(n), z(n)) : n \geq 0\} \quad (3.12)$$

is of dimension  $J$ .

But if  $J > I$ , this set has dimension  $I$ . This has to be true because, for any given vector  $x$ , any consumer would choose the largest  $z$  with  $(x, z)$  in  $B$ . Therefore  $B$  may be restricted to vectors  $(x, z)$  with a unique  $z$  for each  $x$ , and that is a set of dimension  $I$ . It can be confirmed that (3.1) implies  $(x(m), z(m)) = (x(n), z(n))$  for all  $m$  such that

$$s(x(n), z(n), m) = s(x(n), z(n), n) \quad (3.13)$$



so that a  $(J - I)$ -dimensional set of consumers chooses each point of  $B$ . To prove this, one uses the symmetry of the matrix  $v'' - u_{nn}$ , which is equal to

$$v'' - u_{nn} = u_z \sum_i \frac{\partial s_i}{\partial n_j} \frac{\partial x_i}{\partial n_k} \quad (3.14)$$

The details are left to the reader.

When  $J < I$ , the budget set  $B$  can be extended to an  $I$ -dimensional set without introducing new consumption vectors that any consumer would wish to choose. Then it can be expressed in the form

$$z = c(x). \quad (3.15)$$

In the case  $J = 1$ , where  $B$  is 1-dimensional, it is often possible to describe the optimal budget set in the 1-dimensional form

$$x_i = a_i(z), \quad i = 1, \dots, I, \quad (3.16)$$

which may be preferable administratively. This requires, of course, that  $x_i$  and  $z$  are monotonically related in the optimum. It is only when  $J > 1$  that some cross-dependence of tax rates is generally necessary for optimality.

The most important aspect of the conditions developed in this section, from the point of view of the previous section, is that the attractive simplicity of the marginal-tax-rate equation (2.19) has to some extent been lost in its generalized form (3.7). The signs of marginal tax rates can no longer be determined without prior knowledge of the multipliers  $\mu_j$ . Correspondingly, the conditions for Pareto-efficiency come from

$$\sum_j \mu_j \frac{\partial}{\partial n_j} (\tau_{i_1} / \tau_{i_2}) = 0 \quad (3.17)$$

and say that the  $J \times (I - 1)$  matrix  $[\frac{\partial}{\partial n_j} (\tau_i/\tau_1)]$  has rank no greater than  $J - 1$ . This condition is empty when  $J \geq I$ . A principal purpose of this section has been to warn that the results of the previous section depend for their relative simplicity on the one-parameter assumption. At the same time, it should be remembered that only models with a small number of parameters are likely to be of any use for the practical implementation of optimal tax theory.

#### 4 Optimal Mixed Taxation

The results of the nonlinear approach to the theory of optimal taxation are interesting and promising. What should be emphasized is the contrast with the linear results, that the nonlinear conditions say some rather clear things about tax rates, whereas the linear conditions say something about demand changes. It is hard to resist the appeal of conditions where tax rates appear explicitly. But this is achieved at considerable cost. The form of budget constraint the government is supposed to be able to impose is extremely general, allowing progression or regression in the taxation of all commodities, at rates which can depend upon the consumption of different commodities. That is to say, the satisfaction of the first-order conditions must be expected to be inconsistent with a simple tax system, with constant tax rates, or even tax rates dependent upon the consumption of the commodity taxed alone. What worries me about this is not the difficulty of persuading government to adopt complicated tax systems, but the serious neglect of tax avoidance possibilities (by trade among consumers) which the comparison of these very general tax systems commits us to. In this section, I derive conditions for an optimal

tax system in which some commodities (i.e., one at least) are subjected to nonlinear taxation, while the others are subject to constant tax rates. Thus, in particular, nonlinear taxation can be restricted to commodities in which retrading is impossible or perfectly observed.

The notation to be used in this section is:

$x$ : vector of net demands for commodities subject to proportional taxation

$p$ : Producer prices for these commodities

$q$ : consumer prices for these commodities

$z$ : vector of net demands for commodities subject to nonlinear taxation

$r$ : producer prices for these commodities.

The population is described by a single parameter  $n$ .

Define an expenditure function

$$m(q, z, v, n) = \text{Min}[q \cdot x : u(x, z, n) = v]. \quad (4.1)$$

Then we can define a compensated demand function, in this case for the goods subject to proportional taxation only, with demand for the other goods given:

$$x^c(q, z, v, n) = m_q(q, z, v, n). \quad (4.2)$$

We are contemplating allocations in which, for each  $n$ ,

$$x(n), z(n) \text{ maximize } u(x, z, n) \text{ subject to } (q \cdot x, z) \in B, \quad (4.3)$$

where  $B$  expresses the nonlinear taxation. For example, in the case where one good only is subject to nonlinear taxation,  $B$  would be described by  $q \cdot x \leq wz - t(z)$ ,  $w$  being the price of that good.

As in the case of completely nonlinear taxation, we want to express (4.3) in a more readily manipulated form. To this end, define a partially indirect utility function

$$u^*(q, y, z, n) = \text{Max}\{u(x, z, n) : q \cdot x \leq y\} \quad (4.4)$$

(4.3) is equivalent to

$$y(n) = q \cdot x(n), z(n) \text{ maximize } u^*(q, y, z, n) \text{ subject to } (y, z) \in B \quad (4.5)$$

$$x(n) \text{ maximizes } u(x, z(n), n) \text{ subject to } q \cdot xy(n) \quad (4.6)$$

We know already from Section 2, and the appendix to the paper, that

(4.5) is equivalent to

$$v'(n) = u_n^*(q, y(n), z(n), n) \quad (4.7)$$

$$\text{where } v(n) = u^*(q, y(n), z(n), n) \quad (4.8)$$

along with inequality constraints which can be neglected in a large class of cases. (4.6) is equivalent to

$$x(n) = x^c(q, z(n), v(n), n), \quad y(n) = m(q, z(n), v(n), n) \quad (4.9)$$

As a result of these transformations, we address ourselves to the following problem

$$\text{Max } \int v(n) f(n) dn \text{ subject to } \int [p \cdot x^c(q, z, v, n) + r \cdot z] f(n) dn = A \quad (4.10)$$

$$v'(n) = u_n^*(q, m(q, z, v, n), z, n)$$

From this simple form, the first-order conditions for optimal mixed taxation are readily derived. The Lagrangean is

$$\begin{aligned} L &= \int \{ [v - \lambda(p \cdot x^c + r \cdot z)] f + \mu v' - \mu u_n^*(q, m, z, n) \} dn \\ &= \int \{ [v - \lambda(p \cdot x^c + r \cdot z)] f - \mu' v - \mu u_n^* \} dn - \mu(0)v(0) + \mu(\infty)v(\infty) \end{aligned} \quad (4.11)$$

The derivatives of  $L$  with respect to  $q$ ,  $v(\cdot)$ , and  $z(\cdot)$  are to be set equal to zero.

Optimizing first with respect to  $q$ , we obtain conditions for optimal commodity taxation:

$$\lambda \int p \cdot x_q^c \text{fdn} + \int \mu (u_{nq}^* + u_{ny}^* m_q) \text{dn} = 0 \quad (4.12)$$

As usual, Slutsky symmetry implies that

$$p \cdot x_q^c = (q - t) \cdot x^c = -t \cdot x_q^c = -x_q^c \cdot t \quad (4.13)$$

To evaluate  $u_{nq}^* + u_{ny}^* m_q$ , we use the fact (Roy's theorem for the situation) that

$$u_q^* + u_y^* x(q, y, z, n) = 0 \quad (4.14)$$

where  $x(q, y, z, n)$  is defined as the  $x$  that maximizes  $u(x, z, n)$  subject to  $q \cdot x \leq y$ . (4.14) is an identity. Differentiating partially with respect to  $n$ , we get

$$u_{nq}^* + u_{ny}^* x + u_y^* x_n = 0 \quad (4.15)$$

Substitution of (4.13) and (4.15) into (4.12) yields

$$(\int x_q^c \text{fdn}) \cdot t = - \int v(n) x_n(q, y(n), z(n), n) \text{dn} \quad (4.16)$$

$$\text{where} \quad v(n) = u_y^* \mu / \lambda \quad (4.17)$$

In Section 1, it was argued that  $-(\int x_q^c \text{fdn}) \cdot t$  is a satisfactory measure of the extent to which commodity taxes discourage consumption of the different commodities. (4.16) says that discouragement should be zero when  $x_n = 0$ . We shall see that  $v$  is normally nonnegative

in such cases as the model with  $n$  interpreted as ability. Then  $x_{in} > 0$  implies that commodity  $i$  should be discouraged, while  $x_{in} < 0$  implies that it should be encouraged.

This surprisingly simple criterion says that commodity taxes should bear more heavily on the commodities high- $n$  individuals have relatively strongest tastes for. Notice that the criterion looks at the way in which demands change for given income and labour supply when  $n$  changes. The spirit of the criterion is more akin to the two-class criterion (Mirrlees (1975)) than to that of Ramsey (1927). But it is surprising, and satisfactory, that it is expressed as an integral of  $\partial x / \partial n$  rather than an integral of  $x$ .

Turning to the conditions for nonlinear taxation, we differentiate L (4.11) with respect to  $z(n)$  and  $v(n)$ :

$$-(p \cdot x_z^c + r)\lambda f = \mu(u_{ny}^* m_z + u_{nz}^*) \quad (4.18)$$

$$\mu' + u_{ny}^* m_v \mu = (1 - \lambda p \cdot x_v^c) f \quad (4.19)$$

$$\mu(0) = 0, \quad \mu(\infty) = 0 \quad (4.20)$$

These equations correspond in a general way to (2.17), (2.16) and (2.18) above. The correspondence can be brought out more clearly if we define the marginal rate of substitution between  $y$  and  $z$ :

$$s = u_z^* / u_y^* = - m_x \quad (4.21)$$

$$\text{Then } s_n = u_{nz}^* / u_y^* - s u_{ny}^* / u_y^* = (u_{nz}^* + u_{ny}^* m_z) / u_y^* \quad (4.22)$$

$$\begin{aligned} \text{and } p \cdot x_z^c &= q \cdot x_z^c - t \cdot x_z^c = m_z - t \cdot x_z - t \cdot x_y m_z \\ &= - (1 - t \cdot x_y) x - t \cdot x_z \end{aligned} \quad (4.23)$$

Note also that

$$m_v u_y^* = 1 \quad (4.24)$$

$$\begin{aligned} \text{and} \quad p \cdot x_v^c &= q \cdot x_v^c - t \cdot x_v^c = m_v - t \cdot x_y m_v \\ &= (1 - t \cdot x_y) / u_y^* \end{aligned} \quad (4.25)$$

Introducing the results of these calculations into (4.18) and (4.19), and using definition (4.17), we obtain

$$[(1 - t \cdot x_y)s + t \cdot x_z - r]f = v s_n \quad (4.26)$$

$$\mu' + \frac{u_{ny}^*}{u_y^*} \mu = [1 - \frac{\lambda}{u_y^*} (1 - t \cdot x_y)]f \quad (4.27)$$

In order better to appreciate these equations, it should be recognized that  $(1 - t \cdot x_y)s + t \cdot x_z - r$  is the total marginal tax rates on the  $z$  commodities, including both linear and nonlinear taxes.

When applying the theory of optimal income taxation to an economy with many commodities, one would use a utility function for disposable income and labour which assumes given prices for commodities. Both in actual economies and optimized ones, these prices are not proportional to the social marginal costs of commodities. (4.26) and (4.27) show how one should allow for the differences. Calculations of optimal tax rates that have been done (e.g., Mirrlees (1971) and Stern (1975)) do not allow for this, so it is interesting to consider in which direction the results have been biased, and whether these calculated optimal rates are closer to the optimal income tax or the optimal income tax or the optimal total tax rate (including commodity tax effects).

The interrelations of the various equations are so complicated that no general answer to these questions seems possible. But it is instructive to consider the special case



$$u = U(a(x,n) + b(z,n)). \quad (4.28)$$

It will be recollected that, when  $a_n \equiv 0$ , it is optimal to have no commodity taxes (linear or nonlinear), so long as it is possible to tax the  $z$ -goods nonlinearly (Section 2 above). That case is therefore of no interest for our present purpose. Instead take the case where

$$a \text{ is homogeneous of degree one in } x \quad (4.29)$$

It is then easily seen that the functions  $s(q,y,z,n)$  and  $x(q,y,z,n)$  used in the theory above have the following properties:

$$s_y = 0 \quad (4.30)$$

$$x_y = x/y \quad (4.31)$$

$$x_z = 0 \quad (4.32)$$

Also, in indirect form,  $a = ya^*(q,n)$ .

We need to know how  $t \cdot x_y$  varies with  $n$ :

$$\begin{aligned} \frac{d}{dn} t \cdot x_y &= \frac{d}{dn} \frac{t \cdot x}{y} = \frac{t \cdot x_n + t \cdot x_y y' + t \cdot x_z \cdot z'}{y} - \frac{t \cdot x}{y^2} y' \\ &= \frac{t \cdot x_n}{y} + \frac{t \cdot x}{y^2} y' - \frac{t \cdot x}{y^2} y' \\ &= \frac{t \cdot x_n}{y} \end{aligned} \quad (4.33)$$

Now from (4.16) we have

$$\begin{aligned} \int v t \cdot x_n dn &= - \int t \cdot x_q^c \cdot t f dn \\ &\geq 0, \text{ by Slutsky} \end{aligned} \quad (4.34)$$

This suggests that, normally,

$$v \frac{d}{dn} t \cdot x_y \geq 0 \quad (4.35)$$

Trying to confirm that  $v$  is nonnegative, we use the computation (following the argument that led to (2.32))

$$\begin{aligned} \frac{d}{dn} u_y^* &= u_{ny}^* + U''(ya^* + b)(y'a^* + b_z z')a^* \\ &= u_{ny}^*, \end{aligned} \quad (4.36)$$

since  $y' = -s \cdot z' = -(b_z/a^*) \cdot z'$ . Also, note that

$$u_{ny}^* = U'' \cdot (ya_n^* + b_n)a^* + U' \cdot a_n^* \quad (4.37)$$

A variety of assumptions would make the left hand side of (4.37) nonpositive: for example, it is sufficient that

$$\frac{zU''(z)}{U'(z)} \leq -1, \quad \frac{\partial}{\partial n} (a^*/b) \leq 0 \quad (4.37)$$

Under these assumptions

$$\frac{d}{dn} u_y^* \leq 0 \quad (4.38)$$

We also have (from (4.36), and the definition of  $v$ , (4.17))

$$\frac{d}{dn} v = \left( \frac{u_y^*}{\lambda} - 1 + t \cdot x_y \right) f \quad (4.39)$$

I shall show that, if (4.35) holds,  $v \geq 0$  for all  $n$ . Suppose that on the contrary,  $v(n_1) < 0$ . Since  $v(0) = 0$ , there exists  $n_2$  such that  $v(n_2) < 0$  and  $v'(n_2) < 0$ . Then, by (4.39), (4.35), and (4.38),

$$\frac{d}{dn} \left( \frac{1}{f} \frac{dv}{dn} \right) \leq 0 \quad \text{in a neighborhood of } n_2.$$

Consequently,  $v'$  is nonincreasing in the neighbourhood of  $n_2$ , and  $v$  decreases. This must hold, then, for all  $n \geq n_2$ , and we conclude that

$$v(n) < v(n_2) < 0 \quad \text{for all } n > n_2 \quad (4.40)$$

This contradicts the condition that  $v \rightarrow 0$  as  $n \rightarrow \infty$ .

Collecting arguments, we have shown that, on the reasonable postulate that (4.35) holds,

$$v \geq 0 \quad (\text{all } n) \quad (4.41)$$

$$\frac{d}{dn} (t \cdot x_y) \geq 0 \quad (\text{all } n) \quad (4.42)$$

Thus the introduction of commodity taxes introduces an increasing term into the right hand side of equation (4.39). Since  $v$  begins and ends at zero, this seems to mean that  $v$  should increase more slowly and then decrease more slowly than in the absence of commodity taxes: that is,  $v$  should be smaller.

The consequences of this for tax rates may be understood most clearly for the case of a simple income tax, where labour is the only commodity subject to nonlinear taxation. Taking labour as numéraire,  $r = 1$ ; and  $-z$  is labour supplied ( $z$  being a negative number). Let the income tax be  $\theta(-z)$ . Then the budget constraint is  $y = -z - \theta(-z)$ , and the consumer equates  $s = u_z/u_y$  to  $1 - \theta'(-z)$ . Thus the right hand side of (4.26),

$$\begin{aligned} & (1 - t \cdot x_y)s + t \cdot x_z - r \\ &= (1 - t \cdot x_y)(1 - \theta') + t \cdot x_z - 1 \\ &= - \frac{\partial}{\partial(-z)} [t \cdot x(q, -z - \theta, z, n) + \theta(-z)] \\ &= - \text{marginal total tax on labour earnings.} \\ &= - \tau, \text{ say.} \end{aligned}$$

(4.26) says that

$$\tau = -v s_n \quad (4.43)$$

and, if  $n$  represents ability,  $s_n = \frac{\partial}{\partial n} (u_z^*/u_y^*)$  should be negative, since  $n$ -men find it easier to supply labour.

We find, then, that lower  $v$  implies lower  $\tau$ : the tax system as a whole should, it seems, bear less heavily upon labour on the margin than one would suppose if commodity taxes were not allowed for. From the calculations above (noting that  $x_z = 0$  in the special case we are dealing with) we have

$$\theta'(-z) = \frac{\tau}{1 - t \cdot x_y} - \frac{t \cdot x_y}{1 - t \cdot x_y} \quad (4.44)$$

so that  $\theta'$  should be yet smaller than  $\tau$ , if commodity taxes are predominantly positive.

In the above argument it was, first, assumed that individuals had a special form of utility function; and, second, an unproved (but reasonable) conjecture was made at (4.35). It would be even harder to push through a corresponding argument for the general case. Yet the argument provides some reason for believing that the usual calculations of income tax rates are too high, not only because the commodity tax system does and should do some of the work of the income tax; but also because the varying pattern of demand with ability normally requires a lower marginal total tax on labour income. It would be interesting to know whether calculations for plausible cases would support this conjecture. Though strongly suggested by the equations, it is hard to understand intuitively.

It is worth noting one further feature of the first-order conditions, namely that they contain within them some necessary conditions for Pareto-

efficiency, which we can obtain by eliminating  $v$  from equations (4.16) and (4.26):

$$- (\int x_q^c f_{dn}) \cdot t = \int \{ (1 - t \cdot x_y) s + t \cdot x_z - r \} s_n^{-1} x_n f_{dn} \quad (4.45)$$

These relations are not easy to interpret as they stand. The main point is that the weighting function  $v(n)$  appearing in the conditions for commodity taxation and for income taxation must be the same. (4.45) could allow one to calculate optimal commodity tax systems once the optimal income tax was known, without further reference to the welfare function. It might be more interesting to use them as a means of identifying commodities which are affected in a grossly nonoptimal way by the tax system. But in this form, the efficiency conditions seem not much more than a theoretical curiosity.

This completes the discussion of mixed taxation. It is interesting to note that the optimality conditions (4.16), (4.26) and (4.27) contain within them both the conditions for optimality where full nonlinearity is possible, and the conditions where only linearity is possible. The theory of linear taxation is obtained simply by omitting  $z$  and ignoring (4.26). In this case  $x_n = \frac{d}{dn} x$ , and the right hand side of (4.16) can be integrated by parts. The details will be left to the reader. The conditions for nonlinear taxation are obtained by letting  $x$  be one-dimensional. No control is then lost by setting  $q = p$ . Then (4.16) is irrelevant, and the remaining equations reduce to the theory of Section 3.

## 5 Public Goods in an Optimal Mixed System

The point to be made in this section is a simple one. Let  $g$  be a public good, entering all utility functions (possibly trivially in

some cases) and with production price  $\pi$ . In the Lagrangean (4.11),  $x_g^C$ ,  $u_n^*$ , and  $m$  are now all functions of  $g$  as well as the other variables; and  $\pi g$  appears as an additional term in the production constraint, so that one subtracts  $\lambda \pi g$  from the Lagrangean. Then differentiation with respect to  $g$  yields

$$\lambda \int p \cdot x_g^C f dn + \lambda \pi + \int \mu (u_{ng}^* + u_{ny}^* m_g) dn = 0 \quad (5.1)$$

By means of the usual manipulation in tax theory, we have

$$\begin{aligned} p \cdot x_g^C &= (q - t) \cdot x_g^C \\ &= m_g - t \cdot (x_g + x_y m_g) \\ &= -t \cdot x_g + (1 - t \cdot x_y) m_g \end{aligned} \quad (5.2)$$

Furthermore,  $m_g = -u_g^*/u_y^*$ , and we want to define

$$\sigma = u_g^*/u_y^* = -m_g \quad (5.3)$$

as the marginal rate of substitution between income and the public good.

Partial differentiation of (5.3) with respect to  $n$  gives

$$\begin{aligned} \sigma_n &= (u_{ng}^* - u_g^* u_{ny}^* / u_y^*) / u_y^* \\ &= (u_{ng}^* + u_{ny}^* m_g) / u_y^* \end{aligned} \quad (5.4)$$

Using (5.2) and (5.4) we can write (5.1) in the form

$$\pi = \int (1 - t \cdot x_y) \sigma f dn + \int t \cdot x_g f dn - \int v \sigma_n dn \quad (5.5)$$

This says that the supply of public goods should be at such a level that their marginal costs are equal to a sum of

- (i) individual marginal rates of substitution for the public goods, reduced by a proportion equal to the derivative of commodity taxes with respect to income spent on them (for that individual),
- (ii) the derivatives of total commodity tax revenue with respect to the provision of the public goods, less
- (iii) a weighted sum of terms expressing how the personal value of the public good varies with  $n$ .

The first two parts of this expression give a direct estimate of the social value of the good, adding marginal rates of substitution in the usual way, and making allowance for direct revenue effects. The last part of the expression corrects this estimate for distributional considerations. It uses the fundamental weighting factors  $v(n)$  which played such an important part in the conditions for optimal taxation, and applies them to the quantities  $\sigma_n$ , which show how much preference for the public good varies with  $n$ . This is analogous to the way in which  $v$  multiplied  $s_n$  and  $x_n$  in the optimal tax conditions. In the normal case with  $v \geq 0$ , the correcting term is negative if the more able have a stronger preference for the public good, positive if the less able have. Thus the rule encourages provision of public goods valued by the poor and discourages those valued by the rich. The existence of optimal taxes does not eliminate distributional considerations in the provision of public goods; but it does allow the distributional considerations to be analysed out and expressed as a separate contribution to (or deduction from) the marginal social value of the good.

It will be recognized that, in the one-parameter case,  $v$  can be deduced from the optimal taxes themselves through (4.26), and substituted



in (5.5). Thus we obtain a further set of welfare-independent conditions which are necessary for Pareto efficiency. In the case of zero commodity taxes (or fully nonlinear taxation) these conditions take a rather neat form:

$$\pi = \int (\sigma - \tau_i \frac{\sigma_n}{s_{in}}) f dn \quad (5.6)$$

where  $\tau_i$ , the marginal tax on commodity  $i$ , is  $s_i - p_i$ . An alternative form is

$$\pi = \int \sigma (1 - \frac{\sigma_n / \sigma}{\tau_{in} / \tau_i}) f dn \quad (5.7)$$

In this form it is plain that the social value of the public good is enhanced when its value for some individual responds more sensitively to  $n$  than do his marginal tax rates.

## 6 The Fundamental Principle of Optimal Tax Theory

We have derived necessary conditions for the optimality of various economic policy variables. These conditions have an essentially simple common structure, which it is the purpose of this section to emphasize.

If an economic policy is optimal, there is no change in it which will leave total welfare unchanged and at the same time generate a net addition to government revenue. Consider a small change in some policy,  $\delta P$ , and associate with it income transfers to and from individuals which are designed to keep total welfare constant. These transfers may be analyzed into two parts. First an income transfer  $\delta y_1$  made that would, if behaviour did not change, leave each individual's utility unchanged:

this transfer is pure compensation for the initial policy change.

Since it is not in general possible to make these  $\delta y_1$  transfers in a lump sum manner, individuals would want to change their behaviour to benefit from the structure of these transfers. We therefore institute a second set of income transfers  $\delta y_2$  whose effect is to leave total utility unchanged, while removing the incentive to change behaviour arising from the first round of transfers. Specifically, we know that individuals always adjust their behaviour in such a way that  $v' = u_n$ . Having chosen  $\delta y_1$  so that  $\delta v = 0$ ,  $\delta y_2$  is now chosen so that  $\delta v' = \delta u_n$ .

These income transfers create changes  $\delta y_1 + \delta y_2$  in government expenditures, which are partly offset by changes in tax revenue,  $\delta t_1$  arising from the policy change  $\delta P$ , utility being kept constant, and  $\delta t_2$  arising from the second-round income transfers  $\delta y_2$ .

If the policy were originally optimal, the net increase in government revenue does not increase whether the policy change is positive or negative. Thus

$$\int (\delta y_1 - \delta t_1 + \delta y_2 - \delta t_2) f d n = 0 \quad (6.1)$$

$\delta y_1$  and  $\delta t_1$  are easily computed from elementary considerations. Writing

$$\begin{aligned} m(v, P, n) &= \text{income yielding utility } v \text{ for } n\text{-man} \\ &\text{when policy is } P \end{aligned} \quad (6.2)$$

$$\begin{aligned} t(v, P, n) &= \text{tax revenue from } n\text{-man when he has} \\ &\text{utility } v \text{ and policy is } P \end{aligned} \quad (6.3)$$

we have

$$\delta y_1 = \frac{\partial m}{\partial P} \delta P = m_P \delta P \quad (6.4)$$

$$\delta t_1 = \frac{\partial t}{\partial P} \delta P = t_P \delta P \quad (6.5)$$

It remains to estimate  $\delta y_2 - \delta t_2$ . Remarkably, we find that

$$\int (\delta y_2 - \delta t_2) f dn = - \int v(n) \left( \frac{\partial}{\partial n} m_P \right)_{m,P} dn \quad (6.6)$$

where  $m_P$  is differentiated with  $m, P$  held constant; and the multipliers  $v$  (normally nonnegative) are independent of the particular policy considered.

This is what we have been discovering in the previous sections of this paper. There we also found that  $v$  is the solution of the differential equation (rewriting (4.19))

$$\begin{aligned} u_y^* \frac{d}{dn} (v/u_y^*) + u_{ny}^* (v/u_y^*) &= (1 - \lambda_P \cdot x_V^C) f \\ &= (1 - \lambda m_V + \lambda t_V) f, \end{aligned} \quad (6.7)$$

with  $v = 0$  at  $n = 0, \infty$ . The function  $u^*(y, P, n)$  is related to the compensation function  $m$  by

$$v = u^*(m, P, n). \quad (6.8)$$

Thus the fundamental principle is that we can assess the value of a policy by associating with an  $n$ -man three numbers:-

- (i)  $m_P$  = pure compensation for the policy
- (ii)  $- t_P$  = reduction in taxes, at constant utility
- (iii)  $-\frac{v}{f} \left( \frac{\partial}{\partial n} m_P \right)_{m,P}$  = distributional and incentive effects.

The total effect of the policy is measured by adding these numbers for everyone affected. For the three major policy tools examined in this paper we have:-

Policy Instrument	$m_p$	$t_p$
Proportional taxes	$x$	$\frac{\partial}{\partial t} (t \cdot x^c) = x_q^c \cdot t + x$
Public goods	$\sigma = u_g^*/u_y^*$	$\frac{\partial}{\partial g} (t \cdot x^c) = t \cdot x_g^c$
Nonlinear taxes	$s = u_z^*/u_y^*$	$\frac{\partial}{\partial x} (t \cdot x^c + p \cdot z)$

To appreciate the last row of the table, it should be understood that the instruments in the case of nonlinear taxes are best taken to be the demands  $z(n)$  for the taxed commodities (marginal tax rates lead to very complicated equations which can only with considerable labour be reduced to the simple form).

The form in which commodity tax rules appear is also worth attention. For this case the cost of an increase in a commodity tax rate is

$$\int [x - (x_q^c \cdot t + x) - \frac{v}{f} x_n] f dn \quad (6.9)$$

If there are no nonuniform policies (such as nonlinear taxes),  $x_n = \frac{d}{dn} x_n$ , and the last term can be integrated by parts, so that we have the form

$$\int x_q^c \cdot t f dn = \int v'(n) x dn \quad (6.10)$$

which is essentially equations (1.12) of Section 1. In this case  $v'(n)$  is the social marginal utility of income transfer.

In the general case, since  $v$  is the multiplier dual to the constraint

$$\begin{aligned} 0 &= (v' - u_n)/u_y \\ &= m_v v' + m_n \\ &= \left( \frac{\partial}{\partial n} m(v(n), P, n) \right)_p, \end{aligned} \quad (6.11)$$

it is appropriate to call  $v$  the social marginal utility of income-difference.

The striking feature of this analysis is that the distributional aspects of policy choice can be incorporated in a weighting function  $v$ ; which then defines a simple differential operator to be applied to  $m_p$ .  $m_p$  is the old measure of social value, which ignores revenue effects and the desirable 'distortions' arising from distributional considerations. It is easy to see how to supplement it with an estimate of revenue effects; and we now have a simple general principle showing how the remaining adjustment should be made. The method can be applied without difficulty to other types of economic policy, such as quantity rationing, wherever the simple equilibrium model of this paper is thought to be appropriate.

# Appendix

The following lemma will be proved, on the assumption that  $u(x,z,n)$  is a twice differentiable function, increasing in  $z$ ; and  $x,z$  are differentiable functions of  $n$ :

Lemma. If

$$v(n) = u(x(n), z(n), n), \quad v'(n) = u_n(x(n), z(n), n) \quad (A.1)$$

and for all  $m, n$

$$s_m(x(n), z(n), m) \cdot x'(n) \geq 0 \quad (A.2)$$

then there exists  $B$  such that, for all  $n$ ,

$$x(n), z(n) \text{ maximizes } u(x, z, n) \text{ for } (x, z) \in B \quad (A.3)$$

If (A.3) holds, and (A.1) holds and for all  $n$

$$s_n(x(n), z(n), n)' \cdot x'(n) \geq 0. \quad (A.4)$$

Recollect that  $s = u_x/u_z$ .

Proof The second part of the theorem was proved in Section 2 following equation (2.2).

For the first part, suppose on the contrary that there exist  $n_1, n_2$  such that

$$u(x(n_1), z(n_1), n_2) > u(x(n_2), z(n_2), n_2). \quad (A.5)$$

If  $n_1 < n_2$  (a similar argument works if  $n_1 > n_2$ ), (A.5) implies the existence of  $n_3$ ,  $n_1 < n_3 < n_2$  such that

$$\frac{\partial}{\partial n} u(x(n), z(n), n_2) < 0 \text{ at } n = n_3 \quad (A.6)$$

This derivative is equal to

$$\begin{aligned} u_x(x(n_3), z(n_3), n_2) \cdot x'(n_3) + u_z z'(n_3) \\ = u_z [s(x(n_3), z(n_3), n_2) \cdot x'(n_3) + z'(n_3)] \end{aligned}$$

Therefore

$$s(x(n_3), z(n_3), n_2) \cdot x'(n_3) + z'(n_3) < 0 \quad (\text{A.7})$$

But (A.1) implies that

$$s(x(n_3), z(n_3), n_3) \cdot x'(n_3) + z'(n_3) = 0 \quad (\text{A.8})$$

Since  $n_3 < n_2$ , (A.7) and (A.8) imply the existence of  $n_4$ ,  $n_3 < n_4 < n_2$ , such that

$$s_n(x(n_3), z(n_3), n_4) \cdot x'(n_3) < 0, \quad (\text{A.9})$$

contradicting (A.2).

. This proves that (A.3) holds with  $B$  defined as the set of  $(x(n), z(n))$  as  $n$  runs through all values. Thus the Lemma is proved.



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